Random Walks

A random walk is a tuple of $T$ $d$-dimensional steps.

- Random Walk (RW): Single coordinates of steps are drawn from a Gaussian distribution $G(0, 1)$.
- Lattice Random Walk (LRW): Every step moves to an adjacent lattice site. It may only traverse one site per step. Here, the lattice is a square lattice. Immediate reversals are allowed.
- Self-Avoiding Walk (SAW): A LRW but steps may not visit already visited sites.
- Loop-Erased Random Walk (LERW): A LRW but loops are erased from the walk such that there are no intersections.

One-dimensional properties scale like $T^\nu$.

- $1/2 = \nu_{\text{RW}} \leq \nu_{\text{SAW}} \leq \nu_{\text{LERW}}$, equality for $d \geq 4$.

SAW: Model for polymers [1]

- not trivial to generate uniformly distributed instances
- use Markov chain to generate new realizations, i.e., pivot algorithm

LERW: Connected to uniform spanning trees and the Potts model for $Q \to 0$
- intended as an easy version of SAW, but shows different behavior

Large Deviation Simulation

Using sophisticated sampling methods [2], the large deviation tails of the distributions, here with probabilities below $P(L) = 10^{-800}$, can be obtained.

- Markov chain Monte Carlo sampling using a “Temperature” $\Theta$ based Metropolis algorithm
- propose a change of the configuration $c_t \rightarrow c_{t+1}$, accept depending on “energy” difference $\Delta S$ (here: $S$ is either area $A$ or perimeter $L$)

\[ p_{\text{acc}} = \min\left[1, e^{-\Delta S/\Theta}\right] \]

- after equilibration: sample $S$
- biased distributions $P_{\Theta}(S)$ can be transformed into the true distribution

\[ P(S) = e^{S/\Theta} Z(\Theta) P_{\Theta}(S) \]

- calculate $Z(\Theta)$ by matching overlaps
- can be applied to a wide range of problems and enhancements like parallel tempering are applicable [3]
- alternative: Wang Landau sampling

Distributions of Area and Perimeter

Distribution for SAW and LERW over whole support

- decent system sizes up to $T = 2048$
- no small $L$ and $A$ because of excluded volume effects

Whole distribution scales well with $T^\nu$, as in the RW case [4]

- deviations near $S_{\text{max}}$ due to lattice structure
- excellent at the peak region

Empirical rate function

\[ \Phi(S/S_{\text{max}}) = -1/T \log P(S) \]

- extrapolated from the large $S$ tail for $T \to \infty$ by a power law
- shows slight deviations from prediction $\kappa = 1/(1-\nu)$
- fits better for $L$ than for $A$
- $A$ shows stronger effects of the lattice structure

Convex Hulls

The convex hull of a set of points is the smallest convex polygon enclosing all $T$ points.

- area $A$ and perimeter $L$ characterize the point set
- well researched in the context of computer graphics
- fast: $\sim O(T \log T)$
- easy generalization to points in $d$ dimensions

Bibliography