

Phase Transitions of the Typical Algorithmic Complexity of the Random Satisfiability Problem Studied with Linear Programming

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ABSTRACT

Here we study the NP-complete Satisfiability problem for N Boolean variables, in particular K -SAT, for which the Boolean formula has the conjunctive normal form with M clauses and $K < N$ possibly negated variables per clause. Although the worst-case complexity of NP-complete problems is conjectured to be exponential, there exist parametrized random ensembles of problems where solutions can *typically* be found in polynomial time for suitable ranges of the parameter. In fact, random K -SAT, with $\alpha = M/N$ as control parameter, can be solved quickly for small enough values of α . It shows a phase transition between a satisfiable phase and an unsatisfiable phase. For branch and bound algorithms, which operate in the space of feasible Boolean configurations, the empirically hardest problems are located only close to this phase transition. Here we study K -SAT ($K = 3, 4$) and the related optimization problem MAX-SAT by a *linear programming approach*, which is widely used for practical problems and allows for polynomial run time. In contrast to branch and bound it operates outside the space of feasible configurations. On the other hand, finding a solution within polynomial time is not guaranteed. We investigated several variants like including artificial objective functions, so called *cutting-plane approaches*, and a mapping to the NP-complete vertex-cover problem. We observed several *easy-hard transitions*, from where the problems are typically solvable (in polynomial time) using the given algorithms, respectively, to where they are not solvable in polynomial time. For the related vertex-cover problem on random graphs these easy-hard transitions can be identified with structural properties of the graphs, like percolation transitions. For the present random K -SAT problem we have investigated numerous structural properties also exhibiting clear transitions, but they appear not be correlated to the here observed easy-hard transitions. This renders the behaviour of random K -SAT more complex than, e.g., the vertex-cover problem.

Introduction

The *Satisfiability problem* (SAT)¹ is to decide whether some Boolean formula is satisfiable or not, i.e., whether for a given Boolean formula, there is an *assignment* of the variables such that the formula evaluates to “true”. SAT is the most-prominent NP-hard (nondeterministic-polynomial) problem. NP-hard problems^{1,2} are fundamental to computational complexity, since NP-hard means that all problems in NP can be mapped in polynomial time to any of them. Despite much effort no algorithm has been found so far, which is able to solve any NP-hard problem in the *worst case* in polynomial time, leading to the famous *P-NP problem*. Thus, all NP-hard problems are considered so far being *hard*. SAT is in particular *NP-complete*³, which means that given a solution it can be verified in polynomial time and is therefore member of the class of NP problems. Thus, if one day someone found a fast algorithm for SAT (which most researcher believe will never happen), one could efficiently solve *all* problems in NP, especially also all NP-complete problems. Anyway, also to advance knowledge helping to prove that actually no fast algorithm exist, one wants to understand the reason for the apparent computational hardness. One somehow empirical approach is to analyse actual (relatively) hard instances of problems. This has attracted much interest in statistical physics⁴⁻⁶ and correspondingly *random ensembles* of suitably parametrized problem instances and their *typical hardness* have been investigated.

This has been done also for SAT, in particular for a subclass defined as follows: All Boolean formulas can be expressed in *conjunctive normal form* (CNF) which is a disjunction of *clauses*, each being a conjunction of variables or negated variables. Therefore K -SAT, which is a Boolean formula in CNF with K distinct variables per clause, is a commonly scrutinized version of the satisfiability problem. The most prominent random ensemble for SAT is the random K -SAT, where for a set of N variables M clauses are generated randomly. Each clause contains K distinct variables which are chosen randomly, and each variable appears negated with probability 0.5. Interestingly, this problem shows a *phase transition* at some critical value α_s of the density $\alpha = M/N$ ⁷. For large problems at $\alpha < \alpha_s$ almost all problems are satisfiable (also denoted as SAT), above α_s almost all

realizations are unsatisfiable (UNSAT). The occurrence of similar phase transitions has been observed frequently for other random ensembles of NP-complete problems⁴⁻⁷, for which 3-SAT is the prime example^{3,8}. This incited strong interest in the K -SAT problem⁹⁻¹² and many other NP-complete problems¹³⁻¹⁸ among physicists.

For the SAT-UNSAT transition it was found that the hardest realizations are located near this α_s . This can be roughly understood because it is trivial to find some solution for much lower values of α and relatively easy to prove a realization unsatisfiable at high values of α .

While this SAT-UNSAT transition is certainly the most scrutinized in the K -SAT problem, there exist more transitions. For example 3-SAT, where SAT-UNSAT occurs at $\alpha_s \approx 4.26$ ⁹, shows a transition to chaotic behaviour at $\alpha_\chi \approx 3.28$ ¹⁹, i.e., using a continuous time deterministic solver²⁰ the trajectory will find the solution if one exists, but it will show chaotic transient behaviour above this threshold resulting in increasing escape rates from attractors. This leads to a higher computational cost and can therefore be used as a measure of hardness. Furthermore, there exists a clustering transition at $\alpha_c \approx 3.86$ ^{11,21}. This means that here the organization of the space of the exponentially many degenerate solutions changes from one big cluster ($\alpha < \alpha_c$) of solutions which are connected in assignment space to a solution space ($\alpha > \alpha_c$) which is fragmented into many non-connected smaller clusters, one says replica symmetry is broken above this threshold.

Usually algorithms like the *branch and bound approach*, *stochastic search* or *message passing* are used in the statistical-mechanics literature. These algorithms operate in the space of feasible assignments and approach the optimum solution from above. Here “optimum” means that the number of unsatisfied clauses is minimized (in the sense of MAX-SAT), i.e., eventually becomes zero if a satisfying assignment is found. Thus for general minimization problems these algorithms yield upper bounds until the true minimum solution is found. As empirically studying the computational hardness always tells something about a problem in conjunction with a specific algorithm, it is desirable to investigate different algorithms, in particular approaches which differ fundamentally. The operations-research literature often uses *linear programming* (LP)^{22,23}, which operates for combinatorial problems outside the space of feasible solutions, i.e., fundamentally different from the above mentioned algorithms. Suitable enhanced (see below) LP techniques are often used for real-world applications since they are versatile and efficient, which means they run typically in polynomial time. For combinatorial problems, e.g., NP-hard optimization problems, the application of pure LP yields solutions which are not necessarily feasible, which here means non-integer-valued assignments to the variables, but which establish a lower bound on the objective. Thus LP somehow approaches for minimization problems the true feasible and optimum solution from below (In this sense the analog solver of^{19,20} also operates outside the feasible region). Nevertheless, a key observation is that whenever LP gives a feasible solution, it must be the true optimum solution of the combinatorial problem. Because of their complementary properties combinations of LP with other approaches yield powerful methods, such as *branch and cut* algorithms²⁴. Here *cutting planes* (CPs) are employed, which are additional inequalities which decrease LP solution space to help finding feasible solutions.

Although LP, in combination with other techniques, is an (efficient) optimization algorithm for practical optimization problems, there are only few studies aiming at understanding the nature of hardness and detecting phase transitions. To our knowledge, studies of the behaviour of LP have only been conducted for the vertex cover (VC)²⁵ and the travelling salesperson problem (TSP)²⁶. For these problems there exist regions in parameter space, where feasible and optimal solutions can be found in polynomial time. For VC, the LP approach yielded solutions up to the percolation transition of the underlying graph ensemble. Therefore, the problem is *easy* with respect to LP up to the percolation transition, and *hard* beyond. For LP improved with cutting planes, another easy-hard transitions occurs at the point of the onset of replica symmetry breaking¹⁶, which here corresponds to the clustering transition of SAT. Note that this coincides with the percolation threshold for the *leaf-removal core*²⁷. This is the point where one would not reasonably expect easy instances anymore. For TSP the easy-hard transitions detected by the LP approach coincided with structural changes of the optimal tour which can be intuitively understood as increases in hardness. Since for TSP many more cutting planes exist, which are not yet tested, it is conceivable to use this technique to find more and more easy-hard transitions this way and understand them, leading to deeper insight into the problem.

Since K -SAT is the archetypal NP-complete problem, we wanted to extend those promising results of the mentioned previous studies. Although K -SAT is by definition a decision problem and not an optimization problem, as we will show below, we found also some “easy-hard” transitions. Nevertheless, these transitions occur at much lower values of the parameter α than the clustering transition, in contrast to VC on random graphs. This result could be related to the fact that K -SAT shows anyway a richer behaviour¹¹, i.e., several different types of transition, in contrast to VC on random graphs. Anyway, we cannot explain this result in the moment, and think it will motivate further studies which aim at the origin of the different behaviour.

Note that this study does not aim to present faster methods to solve the SAT problem, but rather tries to study it in a fundamental sense, aiming at the question “What makes a problem hard”? We pursue this by applying an approach which is widely-used for many practical combinatorial optimization problems, but less-often studied when considering K -SAT, in particular in the computational complexity and statistical-physics communities. In fact, we will show that the easy-hard transitions with respect to the used LP algorithms happen at rather low values of α . Thus, other more specialized algorithms are preferable for practical K -SAT solving.

Models and Methods

***K*-SAT**

A realization of *K*-SAT consists of a Boolean formula over N variables x_i ($i = 1, \dots, N$). The formula is in conjunctive normal form, i.e., it is a conjunction of M clauses c_j ($j = 1, \dots, M$), where every clause is a disjunction of K literals l_{kj} ($k = 1, \dots, K$). A literal is a variable x_i or its negation \bar{x}_i . In each clause, each variable may appear only once. As an example for $N = 4$, $M = 2$ and $K = 3$ take

$$(\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee x_3 \vee \bar{x}_4). \quad (1)$$

This example is solvable with, e.g., $x_1 = \text{“true”} = 1$ and $x_3 = \text{“false”} = 0$, and arbitrary assignments for the other variables. Note that each clause is satisfiable by $2^K - 1$ out of 2^K possible assignments to the variables. Thus, each clause restricts the space of satisfiable assignments a bit. Clearly, with more clauses per variable, i.e., a higher amount of constraints, it is more probable that a random formula is unsatisfiable. As mentioned before, for random 3-SAT with $N \rightarrow \infty$ there is a critical density $\alpha_s = M/N \approx 4.26^9$ at which a phase transition from satisfiable to unsatisfiable (SAT-UNSAT) happens.

Linear Programming

A linear program (LP) is an optimization problem, which can be expressed by a set of linear constraints and a linear objective function, which should be optimized. There are fast (polynomial-time) algorithms to solve a linear program, e.g., the ellipsoid method²⁸ or interior point methods^{29,30}. However, in many sophisticated solvers the simplex algorithm is used, which typically terminates quickly for real-world problems, despite its exponential worst-case time complexity^{22,23}. Though, as soon as some variables need to be integer valued, this problem gets computationally hard. In fact, *integer linear programming* is an NP-hard problem⁸.

A *K*-SAT realization can be expressed as an integer linear program. Therefore every positive literal x_i is expressed as an integer variable x_i and every negative literal \bar{x}_i as $(1 - x_i)$. Since one or more literals of every clause $c \in C$ need to be true for a satisfying assignment, the corresponding integer linear program contains for each clause the constraint that the sum of the expressions for the included literals must be greater or equal 1. The example from Eq. (1) generates following linear inequalities.

$$\begin{aligned} (1 - x_1) + x_2 + (1 - x_3) &\geq 1 \\ x_1 + x_3 + (1 - x_4) &\geq 1 \end{aligned}$$

Since an LP is an optimization problem but SAT is merely a decision problem, we can choose an arbitrary objective function for which to optimize. The simplest objective function is zero, i.e., no optimization.

$$\begin{aligned} \text{min.} & & & 0 \\ \text{s.t.} & & \sum_{x_i \in c_j} x_i + \sum_{\bar{x}_i \in c_j} (1 - x_i) &\geq 1, & \forall 1 \leq j \leq M \\ & & x_i &\in \{0, 1\}, & \forall 1 \leq i \leq N \end{aligned}$$

The last constraint fixes the variables to integer values. We will relax this constraint to $x_i \in [0, 1]$. This allows us to apply a fast LP algorithm to solve the relaxed problem and introduce a measure of hardness for the problem realization. If the LP relaxation yields a solution consisting of only integer variables, the solution is obtained by a polynomial time method and the corresponding realization is obviously easy to solve.

A drawback is that additionally to the principal degeneracy of the problem, i.e., there are possibly many assignments that satisfy the formula, the relaxation leads to a much higher degeneracy. For example, the assignment of all $x_i = 0.5$ is always a solution of the relaxation. After we performed some simulations for SAT in this way, it became evident that this degeneracy is a major problem for this decision problem, which is not present in the optimization problems studied with this method^{25,26}. This degeneracy leads to different behaviour for slight changes in the algorithm. E.g., primal and dual simplex often lead to different behaviour such that for many instances one version will result in an integer solution while the other does not. We observed a similar behaviour when considering different pricing strategies or a presolve stage to tighten the LP. This analysis would therefore only yield information about these technical details and not about the fundamental problem of *K*-SAT. For example, the presolver of both Gurobi and CPLEX can solve easy instances up to a critical $\alpha_{\text{pre}} = 1.640(1)$, which is the same threshold up to which the *pure-literal* rule (also called *affirmative-negative rule*) which is an integral part of the DPLL^{31,32} search algorithm, can solve *K*-SAT realizations, while without presolve the easy-hard transition occurs at a lower value of α – dependent on technical details of the method. Therefore, we do not present results of LP with zero objective in is study.

Instead, we will introduce artificial objective functions to reduce the degeneracy drastically. Further, the choice of the objective function has an influence on the prevalence of integer solutions. Though note, only linear objective functions enable the efficient linear programming techniques. Therefore, non-linear objectives like $\sum_i x_i(1 - x_i)$, which are minimal if all variables x_i are either 1 or 0, are not admissible and in fact generally NP-hard³³.

One simple way to replace the zero objective is maximizing the sum over all variables (MV)

$$\text{max.} \quad \sum_{i=1}^N x_i,$$

which will on average lift variables which are 0 in the integer solution to larger values like 0.5 and thus suppresses integer solutions typically.

Note that this and other additional objective functions have no influence on whether a formula is satisfiable or not, they are just meant as a tool to reduce the degeneracy of the problem to make it less dependent on details of the algorithm and to facilitate finding integer solutions. Both works out as we will see below. As a third objective we tried maximizing the number of fulfilled literals per clause, which we will call *Satisfaction Multiplicity Maximization* (SMM). This can be achieved with a slightly modified linear program by introducing one new variable z_j per clause counting the number of fulfilled literals of its clause and maximizing the sum over all z_j .

$$\begin{aligned} \text{max.} \quad & \sum_{j=1}^M z_j \\ \text{s.t.} \quad & \sum_{x_i \in c_j} x_i + \sum_{\bar{x}_i \in c_j} (1 - x_i) \geq z_j, & \forall 1 \leq j \leq M \\ & x_i \in \{0, 1\}, & \forall 1 \leq i \leq N \\ & z_j \geq 1, & \forall 1 \leq j \leq M \end{aligned}$$

The new kind of constraint ensures that $z_j \geq 1$, i.e., that every clause contains at least one fulfilled literal, such that the solution assignment satisfies the Boolean formula. This type of additional optimization is similar to MAX-SAT, where one tries to maximize the number of satisfied clauses. For MAX-SAT one would instead enforce $0 \leq z_j \leq 1 \forall j$. We also tried this MAX-SAT approach to solve the K -SAT decision problem. Since this formulation does not mitigate the degeneracy problem, and because we did not observe any better performance than by using the other approaches, we do not show results for this approach here.

The SMM objective is an example for a linear objective with a slight preference for integer valued variables, since assignment of all variables of a clause to 1 or 0 according to their polarity contributes more to the objective function than assignments of non-integer values. For example a variable which appears more often unnegated will on average be assigned more often to value 1. This strongly reduces the degeneracy of the solution of the optimization problem, i.e., many of the solutions where the majority of variables are non-integer, are not optimal under this new objective function. Of course this may still yield non-integer values for some variables, which occur in conflicting clauses.

Note that when using LP, finding integer solutions may be facilitated in principle by adding cutting planes, which are further constraints which are added to the problem during run time dependent on the state of the solution process. This allows, e.g. to add a selected small number of constraints from a set of exponentially many ones, for which it would be infeasible to add them all. This was previously observed also for ensembles of random instances for the vertex cover²⁵ and the TSP²⁶. Nevertheless, for the present study this yields no improvement, i.e., no additional phase transitions could be observed, see below. Thus, we do not describe the cutting-plane approach in this section in detail, but rather just list the the ones we tried and did not lead to an improvement.

Mapping to Vertex Cover

All NP-complete problems, by definition, can be mapped onto each other in polynomial time. Thus, it is reasonable to ask, whether transforming SAT instances to instances of another problem and applying algorithms specifically suited for the other problem changes the performance, as measured by the location of the easy-hard transition. Here we used a classical mapping¹ of SAT to VC. For each K -SAT instance an equivalent graph $G = (V, E)$ is constructed in the following way: The set V of nodes contains one pair of nodes i, \bar{i} for each variable x_i ($i = 1, \dots, N$), which represents the variable and its negation. Furthermore, V contains one node (kj) for each literal l_{kj} ($k = 1, \dots, K, j = 1, \dots, M$) in each clause c_j , respectively. Therefore V contains $2N + KM$ nodes. For the set E of edges, for each clause c_j a complete subgraph of size K is formed by connecting all pairs of "literal nodes" (kj) pairwise which correspond to this clause. Also, each "variable node" i is connected with its corresponding

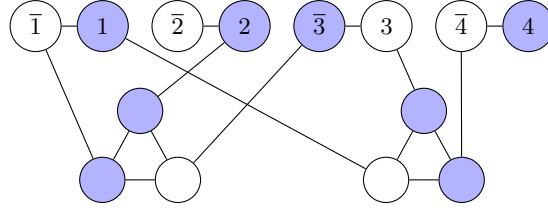


Figure 1. The graph for the vertex cover problem which is equivalent to the formula shown in Eq. (1). Shown is a vertex cover of size $N + (K - 1)M = 4 + 2 \times 2 = 8$, which corresponds to a satisfying assignment $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$.

“negated variable node” \bar{i} . Finally, for each literal l_{kj} , if the literal represents a non-negated variable x_i , an edge connecting (kj) with i is included, while if the literal represents a negated variable \bar{x}_i the corresponding literal node (kj) is connected with \bar{i} . Thus, E contains $MK(K - 1)/2 + N + MK$ edges. Now a minimum vertex cover is obtained. This is a subset $V' \subset V$ of nodes such that for each edge of E at least one of the two endpoints is in V' . By construction¹, G contains a vertex cover of size $N + (K - 1)M$ if and only if the corresponding formula is satisfiable. In Fig. 1 the graph corresponding to the formula from Eq. (1) is shown.

Thus, one approach to SAT is to transform each formula into the equivalent graph and use an existing algorithm for VC to solve it. We applied an LP formulation with cycle cutting planes, see Ref.²⁵ for details. For the previous work, this algorithm was able to solve VC instances in the parameter-space region, where the solutions were contained basically in one cluster, corresponding to the replica-symmetric region³⁴.

Results

We sample random 3-SAT instances, where each clause may contain any variable at most once. For up to 14 system sizes $N \in [128, 524288]$ we simulated $n = 5000$ realizations for 30 to 100 different values of the density α . For comparison, we also performed simulations for 2-SAT and 4-SAT, with a smaller range of sizes, see below. All error estimates are obtained by bootstrap resampling^{35–37}, except for errors of fit parameters shown in the plots, which are *gnuplot*’s asymptotic standard errors corrected according to Ref.³⁶. To solve the LP realizations, the implementation of the dual-simplex algorithms of the commercial optimization library *CPLEX*³⁸ is used. During the research additionally the primal- and dual-simplex implementations of *Gurobi*³⁹ and *lp_solve*⁴⁰ with multiple pricing strategies were used to ensure that the results are independent from the algorithm and the details of the implementation.

LP-Transitions with Objective Function

As mentioned before, we observed that non-trivial objective functions can be used to obtain a result independent from the details of the LP-solver implementation. Though the objective function itself will have an influence on the position of the transition point. An objective which prefers variables to be integer will result in more integer solutions at the same value of α , i.e., yield a transition at a larger value of α .

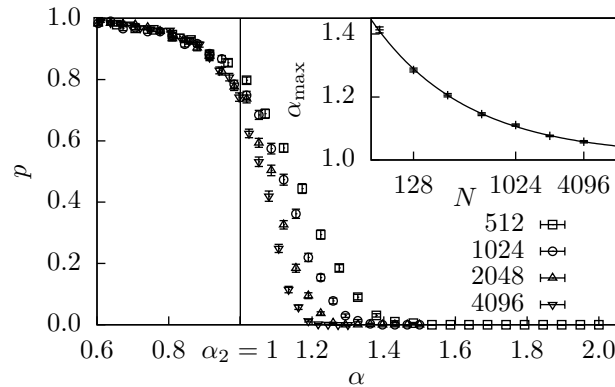


Figure 2. Solution probability p that SMM yields an integer solution for 2-SAT. The inset shows an extrapolation to large N for a transition from solvable to unsolvable. The extrapolation is a power law with offset $\alpha_{\max} = aN^{-b} + \alpha_2$ and yields a transition of $\alpha_2 = 1.011(9)$ with a goodness of fit $\chi_{\text{red}}^2 = 0.6$.

First, we will demonstrate that this method can be used to solve instances of K -SAT efficiently for a range of values of α , and that there exists a phase transition to a hard (or unsolvable) region. For the 2-SAT problem, which is not NP-complete, it is known that the SAT-UNSAT transition happens at $\alpha_2 = 1$ ⁴¹. For 2-SAT there are also exact polynomial time algorithms. So when applying our linear programming approach, we would assume that in the limit of large N for $\alpha < 1$, all instances are easily solvable and for $\alpha > 1$ not. This test does indeed work out when using the SMM objective function. In Fig. 2 the probability that a realization is solvable by the LP+SMM approach, i.e., the solution is integer, is shown as a function of the clause density α . Around $\alpha = 1$ the behaviour switches from solvable to not-solvable. The decrease in probability to solve is steeper for larger system sizes N , which is a behaviour typical for phase transitions. The inset shows the positions of maximum variance, since p is binomial, this corresponds to the positions where half of the realizations are solvable. Therefore, this method effectively extrapolates the position of the $p = 0.5$ point, which should be at the transition point for $N \rightarrow \infty$. We performed the extrapolation by using a typical finite-size scaling power-law ansatz $\alpha_{\max} = aN^{-\tilde{b}} + \alpha_2$, like in previous work^{25,26}. This yielded a critical point of $\alpha_2 = 1.011(9)$, which is in very good agreement with the expectation, especially considering the small system sizes used for this extrapolation. Note that without the objective function 2-SAT is also susceptible to the greater degeneracy and the above mentioned problems become visible, leading to no clear result (not shown here). This result for 2-SAT shows that the method per se is a valid approach to our question.

The same procedure is performed for 3-SAT in Fig. 3. The transition depicted here is an algorithmic transition from *easy*, since most realizations are solvable by LP techniques, i.e., in polynomial time, to some *harder* phase, where the LP does not yield solutions.

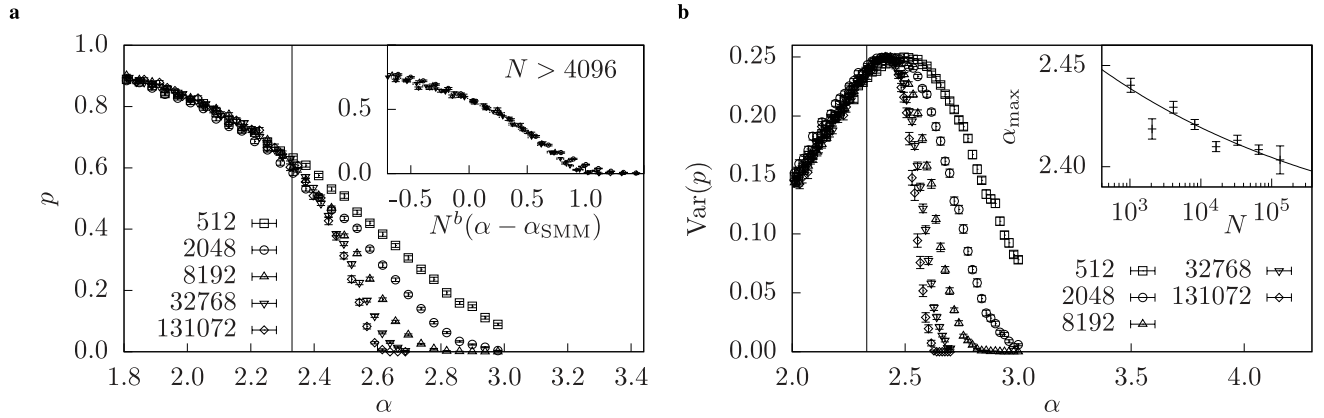


Figure 3. (a) Probability p that SMM yields an integer solution. The smaller system sizes show visible deviations from the common curves, which is visible in the main plot, where $N = 512$ does not lie on the other curves at α_{SMM} marked by the vertical line. Inset: Collapse with $b = 0.118(1)$, $\alpha_{\text{SMM}} = 2.36(1)$ for $N \geq 4096$. (b) Variance of the solution probability p for the SMM objective. Inset: The power-law fit $\alpha_{\max} = aN^{-\tilde{b}} + \alpha_{\text{SMM}}$ to the position of the peaks α_{\max} . For fixed $\tilde{b} = 0.1176$ the fit yields $\alpha_{\text{SMM}} = 2.35(1)$ with $\chi_{\text{red}}^2 = 1.9$.

To estimate α_{SMM} for 3-SAT, we use finite-size scaling, i.e., rescaling the α axis according to $(\alpha - \alpha_{\text{SMM}})N^b$, should result in a *collapse* of the data points on one curve. To find values α_{SMM} and b generating a good collapse the algorithm and implementation from Ref.⁴² is used. It yields $\alpha_{\text{SMM}} = 2.361(7)$ and $b = 0.118(1)$. The collapse obtained when rescaling with those values is shown in the inset of Fig. 2. Note that for second-order phase transitions⁴³ the exponent $\nu = 1/b$ describes the growth of correlations in the system and is one of the *critical exponents* used in statistical physics to characterize phase transitions according to *universality classes*.

We could not use the same extrapolation of the peak positions, like for 2-SAT, since the range of α_{\max} was too small to yield reliable fits. But using the fixed exponent from the collapse, leads to a compatible critical density $\alpha_{\text{SMM}} = 2.35(1)$ and a reasonable fit, with $\chi_{\text{red}}^2 = 1.9$, as shown in Fig. 2.

Our results for 4-SAT look similar (not shown) and exhibit an easy-hard transition as well. We performed a corresponding analysis. The resulting exponent b seems to be larger and a fit through the positions of the maxima yields $\alpha_{\text{SMM}} = 3.60(8)$, $\tilde{b} = 0.5(1)$ with $\chi_{\text{red}}^2 = 0.5$.

The other optimization function of this study, MV, i.e., maximizing the sum of all variables, leads to a qualitatively similar behaviour for 3-SAT as SMM but a transition at lower $\alpha_{\text{MV}} = 1.25(2)$ (not pictured due to qualitative similarity, simulations used smaller system sizes). The lower transition point is plausible, since this maximization prefers variables to be larger than

zero instead of 0. For this reason, we have not analysed this algorithm for 4-SAT. The best estimates for the transition points are collected in Tab. 1.

Table 1. Values of critical points. α_{VC} denotes the critical point when mapping SAT to VC and applying an LP + cutting plane solver used for VC. α_{MV} is the easy-hard transition for LP+MV. α_{SMM} is the easy-hard transition for LP+SMM. α_c denotes the clustering transition¹¹ and α_s the SAT-UNSAT transition^{9,44}.

K	α_{VC}	α_{MV}	α_{SMM}	α_c	α_s
2	–	–	1.011(9)	–	1
3	0.90(3)	1.25(2)	2.361(7)	3.86	4.26
4	–	–	3.60(8)	9.547	9.93

As shortly mentioned above, we also implemented cutting planes (CP), which are additional inequalities which do not change the nature of any solution but restrict the search space of non-feasible, i.e., non-integer configurations. In similar studies on VC²⁵ and the TSP²⁶, the introduction of cutting planes yielded substantially better results. For vertex cover the introduction of (potentially exponentially many but actually few) CPs even lead to an LP+CP transition at the point where in the analytic solution replica symmetry breaking, i.e., clustering of solutions appeared³⁴. Unfortunately, this efficiency of CP was not observable during our study for K -SAT. It seems that the cutting planes we tried, namely *resolution cuts*⁴⁵ and *clique cuts*⁴⁶, were too weak at the low values of α examined here. Another cutting plane for the SAT problem, the *odd cycle inequalities*⁴⁷ are not directly applicable for K -SAT with $K \geq 3$, since they need clauses with 2 variables to be constructed. While they are useful as local cuts in a branch and cut procedure, they are never violated in the beginning for $K \geq 3$ and thus not applicable for this study.

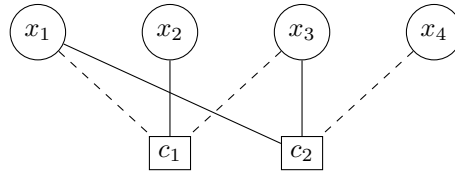


Figure 4. Factor graph representation of the example Eq. (1). Dashed lines represent negative weights, corresponding to negated variables.

Next, we investigate whether the observed easy-hard transitions correspond to changes of the structure of the problem instances, as it was previously found for the vertex-cover problem^{16–18,25}. While VC is defined on graphs anyway, for K -SAT we study the related graph representation as *factor graph*^{5,48}. This representation is especially useful for *belief* or *survey propagation* approaches^{49,50} but also useful to study structural properties. In the FG, which is a bipartite graph exhibiting a node for each variable and a node for each clause, i.e., containing $N + M$ nodes, each variable is connected to the clauses it occurs in with weights representing whether they are negated or not. The corresponding graph of example Eq. (1) is shown in Fig. 4. Note that this representation, when disregarding the weights, is equivalent to a hypergraph representation, where every clause is represented by a hyperedge connecting K variables.

We looked at some known points of the factor graph, where its properties change, which could plausibly influence the hardness of the problem. Note that for the NP-complete VC, it was possible to relate the percolation threshold and the threshold, beyond which the leaf removal heuristic²⁷ does not yield solutions anymore (which coincides with the appearance of replica-symmetry breaking and solution-space clustering), to the points where the problems turns harder also for special formulations of the linear programming approach²⁵. We will list the properties for 3-SAT we looked at. Most transitions are known from the literature, but few not, to our knowledge. For those we numerically investigated the transitions during this study and present the result shortly here, which also contributes to the characterization of the 3-SAT ensemble.

- The *percolation threshold*, i.e., the point below which there will be no connected component of size $\mathcal{O}(N)$ is at $\alpha = 1/6^5$. This means that below this threshold clauses do typically not share variables and are therefore largely independent, i.e., it should be rather easy to solve for almost any algorithm.
- The transition where the remaining *pure literal core* is of order $\mathcal{O}(N)$, i.e., the value of α , beyond which the pure literal rule, which is an integral ingredient for the classical DPLL search, does not lead to solutions anymore is at $\alpha_{pl} = 1.636..^{51}$. The pure literal rule is to remove pure variables and their clauses from the problem. Variables are *pure*, if they are

appearing only in one polarity and can therefore be always set to fulfil all their clauses. Note that for 2-SAT the pure literal rule works up to the SAT-UNSAT threshold $\alpha_2 = 1$ ⁵¹, coinciding with the solvability transition of SMM.

- The *unit clause rule* shows its transition above the LP+SMM case at $\alpha_{uc} = 8/3 \approx 2.66$ ⁵².
- For a naive *q-core* analysis, where the core is the set of remaining nodes after all nodes with degree lower than q are iteratively removed, we treated the clause nodes the same as the variable nodes. It yielded a percolation transition, i.e., the existence of a q -core of order of graph size at $\alpha_{2\text{-core}} = 0.223(3)$ and $\alpha_{3\text{-core}} = 1.554(1)$ from our measurements.
- The *q-core* transitions for a hypergraph approximation of K -SAT, where K nodes are connected by single hyperedges, are known exactly. Note that the 2-core for this ensemble is equivalent to the pure literal rule according to⁵¹. The appearance of a three core occurs at $\alpha_{3\text{-CORE}} = 4.2847..$ ⁵¹.
- The *leaf removal rule*, which is a valid heuristic for the related XOR-SAT problem and the vertex cover problem, is known to have a transition at $\alpha_{lr} = 0.81847..$ ⁵³. Interestingly, although not obviously related, this is half of α_{pl} of the pure literal rule.
- The transition where a *biconnected component*^{23,54} appears, i.e., a connected component, in which every pair of nodes stays in the same connected component if any other node is removed, happens at $\alpha_{bi} = 0.190(20)$. Similarly, the *bi-edge-connected component*, from which an arbitrary edge can be removed while staying connected, shows the transition at $\alpha_{bi\text{-edge}} = 0.211(7)$. This value is close to the appearance of our naive 2-core, which is plausible since a 2-core consists of biconnected components possibly connected by single edges.

The process to determine the transition points for the cases, where we did not find literature values, is very similar to the process shown in Fig. 2. An example for the naive 3-core on the factor graph is shown in Fig. 5. Again we extrapolated the position of the peaks of the variance to large N using a power law with offset $\alpha_{\max} = aN^{-b} + \alpha_{3\text{-core}}$.

When comparing the values of these transitions to the easy-hard transitions points listed in table 1, one observes no coincidence. Thus, in contrast to the previously studied VC, there exist no coincidence with a “simple” change of the problem structure, which might serve as an explanation of the observed easy-hard transitions.

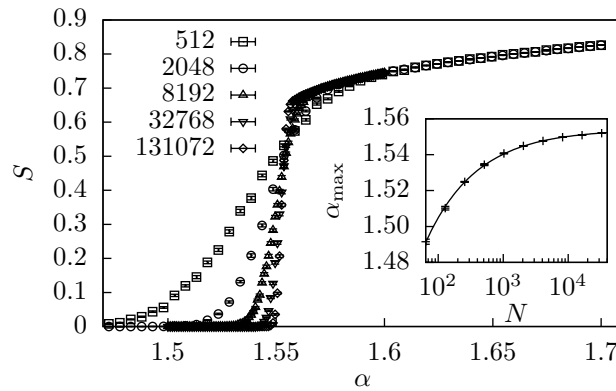


Figure 5. Relative size of the naive 3-core of the factor graph at different values of α . The inset shows the positions of the maxima of the peaks of the variance and an extrapolation according to a power law with offset, resulting in the above listed values.

Finally, the existence of correspondences between easy-hard transitions and structural percolation transitions for the vertex-cover problem motivated us to perform the following test: we also used for $K = 3$ the mapping of K -SAT to VC for formulas up to $N = 10000$. Using LP and cycle cutting planes²⁵ the corresponding equivalent instances VC were solved for various values of α . Again we measured the probability that an instance was solved by an integer solution as a function of α , for different system sizes. Using an analysis (not shown) as for the previous approaches, we were able to extrapolate an easy-hard transition for this point. We obtained a critical value of $\alpha_{VC} = 0.90(3)$, which is well below the easy-hard transitions obtained using the LP-based approaches presented above. Therefore, apparently it does not pay off using a mapping to another problem, at least for this pair of problems. A mapping to TSP will lead to TSP instances which are quite large for decent SAT realizations, therefore we did not perform simulations for this.

It is rather intriguing that SAT behaves so differently in comparison to vertex cover, despite their very close relation. While we can not give the reason for this behaviour, we not only have identified several easy-hard transitions in this study, but have

also shed some light on this apparently fundamental difference. In the future, maybe other approaches can yield a better understanding of this perplexing fact.

Conclusions

We study the solvability of random K -SAT realizations at different values of the clause-to-variable density α using linear programming. A realization is solved if the LP yields an integer solution. Since there are LP algorithms that run in polynomial time, this means such a realization is “easy” in regard to this algorithm.

Since SAT is a decision problem, the application of pure LP actually does not involve optimization, which seems to make the problem less well-behaved. This is probably the reason that also for practical applications LP has not been used widely for SAT and instead methods like the branch-and-bound approach DPLL dominate. Nevertheless, the investigation of the behaviour of any algorithm, even not the most efficient one, might lead to a better understanding of the specific problem structure and of the sources of computational hardness in general. To allow for an investigation of SAT using LP, we included artificial objective functions, which makes the problem numerically better behaved. Note that we also observed that usually more realizations can be solved when including an artificial objective function as compared to the trivial objective function.

We were able to identify several easy-hard transitions, depending on the algorithm which was used. The inclusion of artificial objective functions lead to higher values of the control parameter α , where solutions can still be found. In contrast to previous work on VC on random graphs, none of these transitions coincided with the onset of a clustered solution landscape. The reason might be that K -SAT, or “natural” decision problems in general behave differently. On the other hand, random K -SAT exhibits several transitions for the structure of the solution landscape anyway, in contrast to VC on random graphs. It would be interesting to investigate the relation between LP and the solution landscape more thoroughly in the future.

Anyway, from the practical point of view, we wonder if carefully crafted objective functions could be used to improve the efficiency of solving decision problems with an LP approach, such as branch and cut.

Like for the clustering transitions, we were not able to identify other structural transitions coinciding with an observed easy-hard transition detected by any of the tested LP formulations. Nevertheless, we believe that such a coincidence should exist, since the properties of a problem should be coded in the graph structure. Therefore, this unknown structural property seems to be of a more complex type. The fact that the relation between LP-based easy-hard transitions and structural properties appears to be more complex might also be related to the mentioned richer behaviour of the solution landscape as a function of the density α (in comparison with VC). Thus, here is still some work to be done.

Furthermore, in contrast to the previously studied vertex cover problem and the travelling salesperson problem, we did not observe any improvement by applying cutting planes. This could be due to the type of cutting planes used. Finally, we applied a mapping of SAT to VC and used specific VC LP+cutting plane algorithm, which is able to solve standard Erdős-Rényi VC instances just up to the clustering threshold. Nevertheless, for SAT this did not pay off, the easy-hard transition appears to be at even smaller values of α compared to the MV objective.

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Author contributions statement

A.K.H. conceived the project, R.B. and H.S. developed the simulation. All three authors analysed the results and contributed to the manuscript.

Additional information

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