The entropy of the longest increasing subsequences: typical and extreme sequences

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Outline

The Model: LIS

COunting LIS efficiently

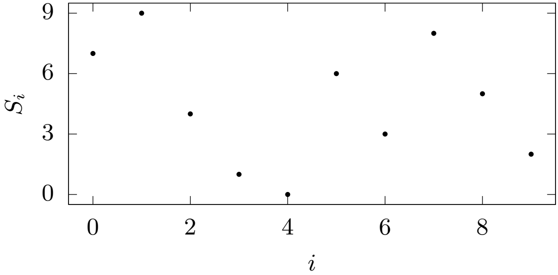
Distribution of the Entropy

Sampling the Far Tails

Longest Increasing Subsequence (LIS)

"Mark the most elements, such that all marked elements left of a marked element are smaller (or equal) than it"

$$7 \quad 9 \quad 4 \quad 1 \quad 0 \quad 6 \quad 3 \quad 8 \quad 5 \quad 2$$



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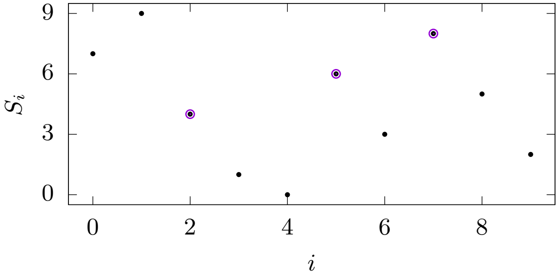

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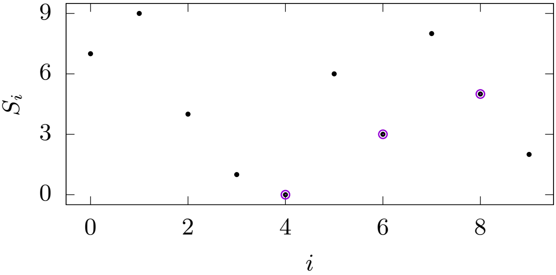

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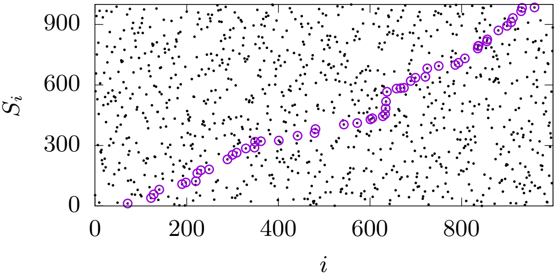

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With a permutation of length n:

- What is the expected length of the LIS? $\Rightarrow 2\sqrt{n}$ [1, 2, 3, 4]
- Why is this interesting?
  - random matrix theory [4]
  - surface growth (KPZ) [5, 6]
  - applications in computer science and bioinformatics
- How many possibilities are there? $\Rightarrow \mathcal{O}(\exp(n))$ [7]

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[1] SM Ulam (1961); [2] K Johansson (1998); [3] J Deuschel, O Zeitouni (1999); [4] J Baik, P Deift, K Johansson (1999); [5] M Prähofer, H Spohn (2000); [6] SN Majumdar, S Nechaev (2004); [7] JM Hammersley (1972)

Counting Longest Increasing Subsequences

Number of different LIS grows exponentially

⇒ We can not just enumerate, we have to count cleverly.

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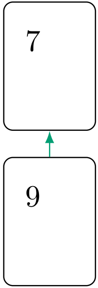

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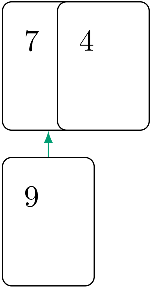

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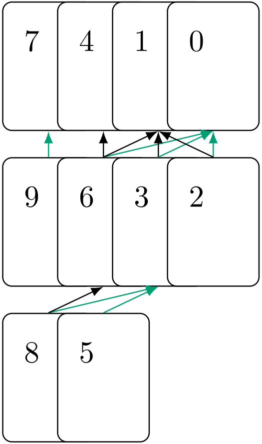

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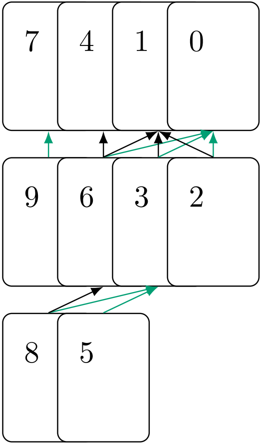

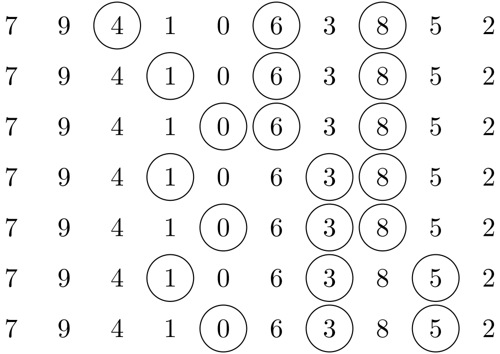

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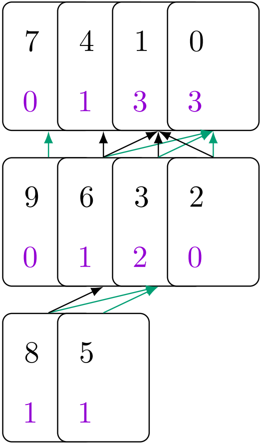
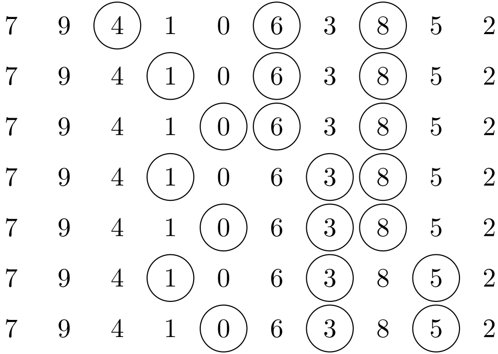
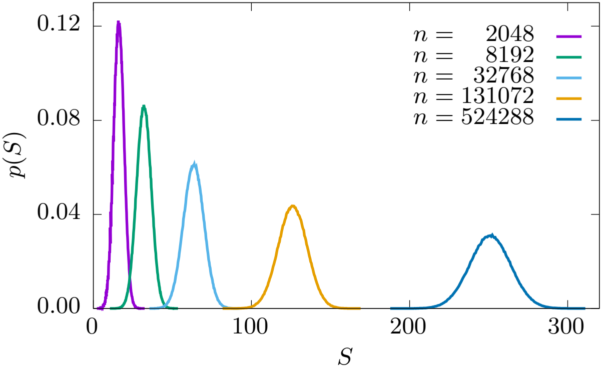




Distribution of the Entropy



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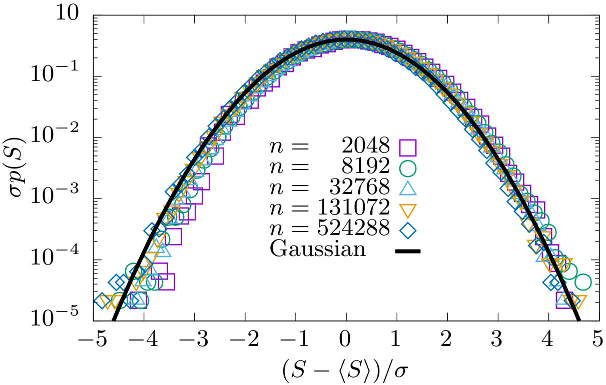


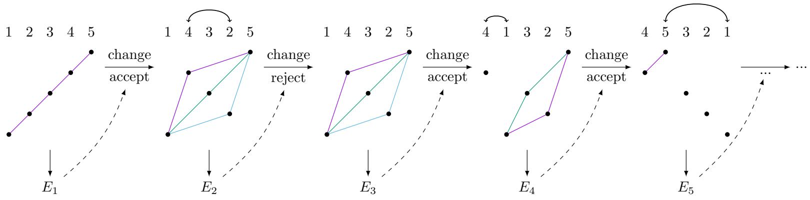
$$\langle S \rangle \approx 0.347\sqrt{n}$$

$$\sigma \approx 0.49\sqrt[4]{n}$$

Markov Chain Monte Carlo for the Far Tails

- treat it as a canonical ensemble, i.e., weights $\sim e^{-E/\Theta}$
- artificial temperature Θ for the disorder ($\boldsymbol{\varepsilon}$)
- Markov chain of realizations $\boldsymbol{\varepsilon} = (\varepsilon_1, .., \varepsilon_N)$
- accept change with probability

$$p_{\mathrm{acc}} = \min\left\{1, e^{-\Delta E_0/\Theta}\right\}$$



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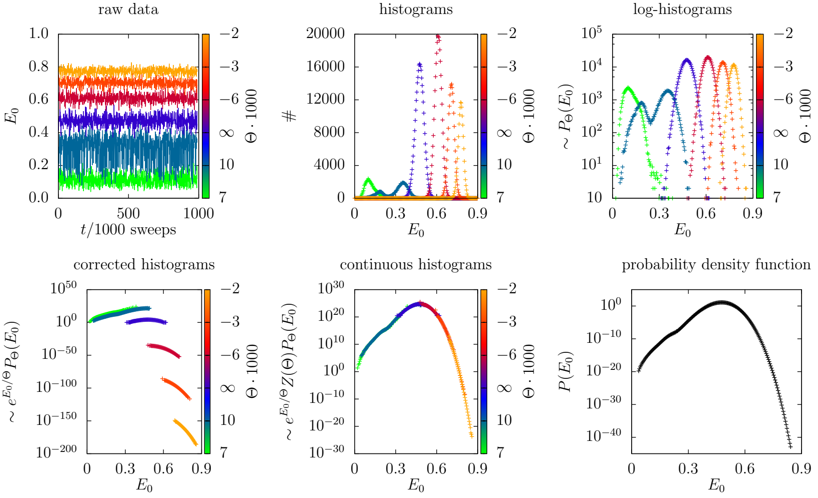
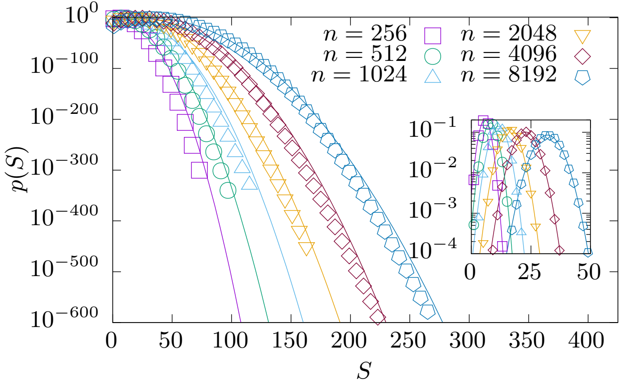
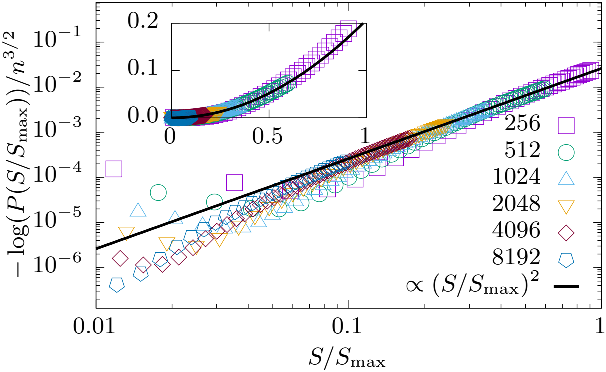


Full Distribution of the Entropy



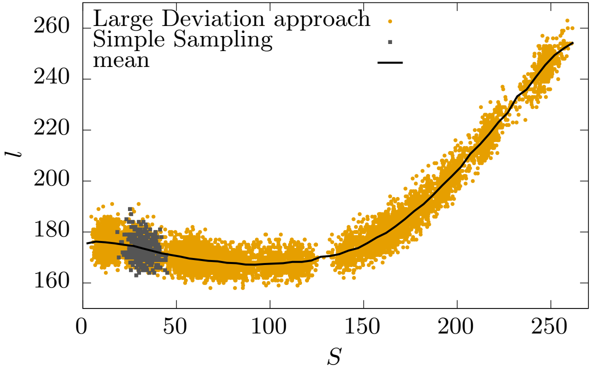
Deviations from Gaussians in the far tail

Full Distribution of the Entropy



Collapse of the far tail onto a rate function $\sim n^2$

Are Length and Entropy correlated?



Conclusions

- For Permutations
  - entropy scales as $S \approx \exp(0.347\sqrt{N})$
  - entropy distribution is well approximated by a Gaussian scaling form
  - far tails decay faster than Gaussian
  - hints for an unusual rate function